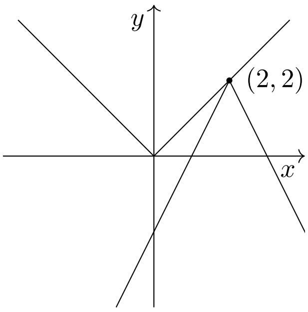
1601. The second graph has its vertex at $4 - 2x = 0$, which gives $(2, 2)$. This is on the graph $y = |x|$. Hence, since the second graph is an inverted mod graph with gradient ± 2, this will be the only point of intersection.



1602. We cannot take the limit as it stands, because both numerator and denominator tend to zero. But, factorising the top, we can cancel a factor of $(x-1)$:

$$\begin{aligned} &\lim_{x\to 1} \frac{x^2-1}{x-1} \\ =\ &\lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} \\ =\ &\lim_{x\to 1} (x+1). \end{aligned}$$

We can now take the limit, which is 2.

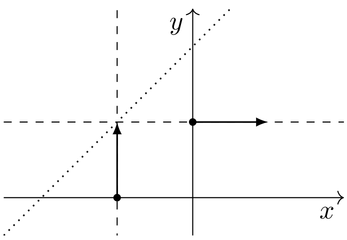
1603. The sine and cosine functions are represented, by definition, as $|OS|$ and $|OC|$. Then the remaining lengths can be found using the fact that every one of the (many!) triangles on the diagram is similar.

$$\begin{aligned} |OS| &= \sin\theta \\ |OC| &= \cos\theta \\ |BP| &= \tfrac{\sin\theta}{\cos\theta} \equiv \tan\theta \\ |AP| &= \tfrac{\cos\theta}{\sin\theta} \equiv \cot\theta \\ |OA| &= \tfrac{1}{\cos\theta} \equiv \sec\theta \\ |OB| &= \tfrac{1}{\sin\theta} \equiv \operatorname{cosec}\theta \end{aligned}$$

1604. (a) If an object is in equilibrium under the action of exactly three forces, then the lines of action of those forces must either all be parallel, or be concurrent. Otherwise, a moment would act around the point of intersection of two of the lines of action. Here, two of the lines of action intersect at $(-1, 1)$. Hence, the third must also.

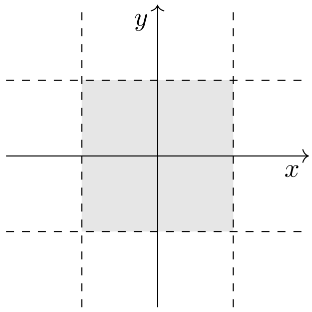
(b) The resultant force is zero, hence $a, b = -2$.

(c) P could lie anywhere on the dotted line:



1605. To find the fixed points of a function f, we solve $\mathrm{f}(x) = x$. Since f is quadratic, this is a quadratic equation. Hence, it has a maximum of two roots. So, no quadratic function can have three distinct fixed points. □

1606. Each region is a corridor between two parallel lines: $x = \pm 2$ and $y = \pm 2$. So, the sketch is



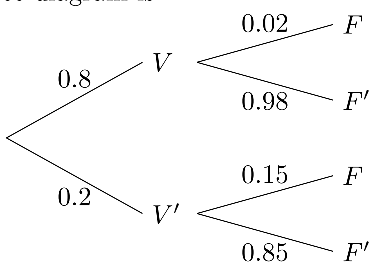
1607. We use the small-angle approximation $\sin x \approx x$, which becomes exact in the limit as $x \to 0$:

(a) $\lim_{x\to 0} \dfrac{x^3}{x^2} = \lim_{x\to 0} x = 0.$

(b) $\lim_{x\to 0} \dfrac{x^3}{x^3} = \lim_{x\to 0} 1 = 1.$

(c) $\lim_{x\to 0} \dfrac{x^3}{x^4} = \lim_{x\to 0} \dfrac{1}{x} = \infty.$

1608. (a) The tree diagram is



(b) The expectations are np in each case:

i. $40 \times \mathbb{P}(F \mid V') = 40 \times 0.15 = 6$ people,

ii. $6 + 160 \times \mathbb{P}(F \mid V) = 9.2$ people.

(c) Restricting the possibility space to F,

$$\mathbb{P}(V \mid F) = \frac{0.8 \times 0.02}{0.2 \times 0.15 + 0.8 \times 0.02} = \frac{8}{23}.$$

So, the ratio is $8 : 15$.

Alternative Method

This ratio is the same as the relevant ratio of expectations: $3.2 : 6$ is equal to $8 : 15$.

1609. Multiplying out and adding, the cross-terms will cancel, so we needn't find them explicitly:

$$\begin{aligned} &\tfrac{1}{2}\Big[\big(\sqrt{x}+\sqrt{x-1}\big)^2 + \big(\sqrt{x}-\sqrt{x-1}\big)^2\Big] \\ \equiv\ &\tfrac{1}{2}\Big[x + 2\sqrt{*} + x - 1 + x - 2\sqrt{*} + x - 1\Big] \\ \equiv\ &\tfrac{1}{2}(4x-2) \\ \equiv\ &2x - 1. \end{aligned}$$

1610. (a) The domain is $[0, \infty)$.

(b) Switching inputs and outputs, we rearrange to make y the subject:

$$\begin{aligned} x &= \frac{\sqrt{y}}{1+\sqrt{y}} \\ \implies x + x\sqrt{y} &= \sqrt{y} \\ \implies x &= \sqrt{y}(1-x) \\ \implies \frac{x}{1-x} &= \sqrt{y} \\ \implies \frac{x^2}{(1-x)^2} &= y. \end{aligned}$$

We can then express the numerator x^2 as $(x-1)^2 + 2x - 1$. Splitting the fraction up, we have the required result

$$\mathrm{f}^{-1} : x \longmapsto 1 + \frac{2x-1}{(1-x)^2}.$$

(c) For 1 to be in the range of f, we would need

$$\begin{aligned} \frac{\sqrt{x}}{1+\sqrt{x}} &= 1 \\ \implies \sqrt{x} &= 1 + \sqrt{x} \\ \implies 0 &= 1. \end{aligned}$$

So, 1 is not in the range of f.

— Alternative Method —

If $y = 1$ were in the range of f, it would be in the domain of f^{-1}. But f^{-1} involves division by $(x-1)$, so is undefined at $x = 1$. Hence, $y = 1$ is not in the range of f.

1611. Longhand, the sum is $x^5 + x^6 + x^7 = 0$, which has a common factor of x^5. The remaining quadratic is $1 + x + x^2 = 0$, which has discriminant $-3 < 0$. Hence, the only root is $x = 0$.

1612. The equations are linear, so the parameters must trisect their respective domains. Considering the endpoints on the diagram, we need $s = 2/3$ and $t = -1$. Substituting, we get $\mathbf{r}_1 = \mathbf{r}_2 = -\mathbf{i} + 2/3\mathbf{j}$. Since the points coincide, the result is verified.

1613. The mean and variance of a binomial distribution are $\mathbb{E}(X) = np$ and $\mathrm{Var}(X) = npq$. So, we have simultaneous equations $np = 30$ and $npq = 22.5$. Dividing gives $q = \frac{3}{4}$, so $p = \frac{1}{4}$ and $n = 120$.

1614. This is not correct. Inextensibility is required to assume that the *accelerations* of both ends are the same. For the *tensions* to be the same, it is only required that ① the rope is light and ② that no friction acts on it, i.e. that any pulleys etc. are smooth. This can be true for an extensible rope.

1615. The area formula gives $12 = \frac{1}{2}5^2 \sin\theta$, where θ lies between the isosceles lengths. Hence, $\sin\theta = \frac{24}{25}$. By the Pythagorean trig identity, this gives two possible values for $\cos\theta$, namely $\pm\frac{7}{25}$. The cosine rule then produces

$$c^2 = 5^2 + 5^2 \mp 2 \cdot 5^2 \tfrac{7}{25} = 36 \text{ or } 64.$$

So, $c = 6$ or 8, as required.

1616. Writing 3 as $9^{\frac{1}{2}}$ and using an index law, we have $3^{2x+4} \equiv 9^{x+2}$. Then, splitting with another index law, $3^{2x+4} \equiv 81 \cdot 9^x$.

1617. This is the area below a $y \geq 0$ semicircle of radius r, centred on O. Hence, the area is $\frac{1}{2}\pi r^2$.

1618. (a) Differentiating, we have $\mathrm{f}'(x) = 3x^2 - 7x^6$. Hence, $\mathrm{f}'(0.8) = 0.084992$.

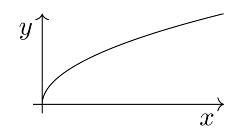
(b) This is a very shallow positive gradient. Hence, when the tangent is drawn, it will give a large, negative value for x_1. In fact, $x_1 \approx -15$. The iteration will then take a long time to work its way back from this point.

1619. (a) Grouping EA as a single "letter", we want the number of rearrangements of five entities: H,EA,L,T,H. There are 5! of these.

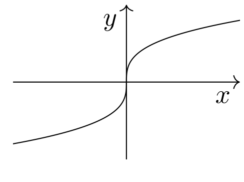
(b) We double the previous answer, to allow for EA and AE. This gives 240.

1620. These nth-root curves look different, depending on the parity (evenness or oddness) of the power. Each is a reflection of the relevant $y = x^n$ curve in $y = x$, but the even powers are restricted to positive inputs.

(a) Curves of the type $y = \sqrt{x}$:



(b) Curves of the type $y = \sqrt[3]{x}$:



1621. Since the circles all have the same unit radius, their being equidistant is the same as their centres being equidistant. The two circles are centred at $(0, 0)$ and $(0, 4)$. The third centre must have coordinates $(x, 2)$ and be a distance 4 away from the origin. Hence, $x^2 + 2^2 = 16$, giving $x = \pm\sqrt{12}$. So, the possible equations are

$$\big(x \pm \sqrt{12}\big)^2 + \big(y - 2\big)^2 = 1.$$

1622. We can write $y = 2^x \equiv (e^{\ln 2})^x \equiv e^{x \ln 2}$. Hence, in transforming $y = e^x \longmapsto y = 2^x$, the x input has been replaced by $x \ln 2$. The transformation is a stretch in the x direction, scale factor $\frac{1}{\ln 2}$. Since this is only a one-dimensional stretch, areas are scaled by the same factor.

1623. In algebra, $x^2 y = k$. By the product rule,

$$\begin{aligned} & x^2 y = k \\ \Longrightarrow\ & 2xy + x^2 \frac{dy}{dx} = 0. \end{aligned}$$

Rearranging this, $\dfrac{dy}{dx} = -\dfrac{2y}{x}$, for $x \neq 0$.

1624. Taking $t = 0$ at projection of the second particle, the times since projection are given by $t+2$ and t. Equating the heights,

$$\begin{aligned} & 20(t+2) - 5(t+2)^2 = 20t - 5t^2 \\ \Longrightarrow\ & 40 - 20t - 20 = 0 \\ \Longrightarrow\ & t = 1 \text{ seconds.} \end{aligned}$$

So, the height is $h = 20 \cdot 1 - 5 \cdot 1^2 = 15$ m.

1625. The implication is $\Longleftrightarrow$: these are two ways of writing exactly the same statement, in logarithm or index language.

1626. (a) In terms of length l, the area is $A = \frac{\sqrt{3}}{4} l^2$. Hence, making l the subject and multiplying by 3, the perimeter is given by

$$P = 2 \cdot 3^{\frac{3}{4}} \cdot \sqrt{A}.$$

(b) To find the rate, we differentiate with respect to t. We need implicit differentiation, i.e. the chain rule on the RHS:

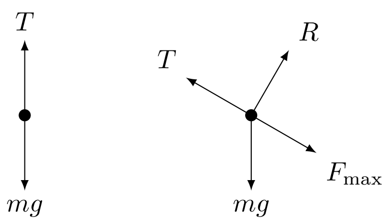
$$\begin{aligned} \tfrac{dP}{dt} &= 2 \cdot 3^{\frac{3}{4}} \cdot \tfrac{1}{2} A^{-\frac{1}{2}} \cdot \tfrac{dA}{dt} \\ &= 3^{\frac{3}{4}} A^{-\frac{1}{2}} \tfrac{dA}{dt}. \end{aligned}$$

Substituting values for A and $\frac{dA}{dt}$ gives

$$\tfrac{dP}{dt} = 3^{\frac{3}{4}} \sqrt{48}^{-\frac{1}{2}} \sqrt{3} = \tfrac{3}{2}, \text{ as required.}$$

1627. It is not well defined. For the boundary values $x = \pm 1$, the square root is well defined as $\sqrt{1-1}$, but this then gives division by zero. The domain should be reduced to $(-1, 1)$.

1628. Force diagrams, in which the acceleration is zero, are as follows. This is limiting equilibrium, with $F_{\text{max}} = \mu R$ acting down the slope.



Resolving for equilibrium perpendicular to the slope, $R = \frac{\sqrt{3}}{2} mg$, so $F_{\text{max}} = \mu \frac{\sqrt{3}}{2} mg$. We can now resolve along the string for the system:

$$\begin{aligned} & mg - \mu \tfrac{\sqrt{3}}{2} mg = 0 \\ \Longrightarrow\ & \mu = \tfrac{2}{\sqrt{3}}. \end{aligned}$$

So, we require $\mu \in \left[\frac{2}{\sqrt{3}}, \infty\right)$.

1629. The chain rule gives

$$\begin{aligned} \frac{dy}{dx} &= 2 \tan 3x \cdot \sec^2 3x \cdot 3 \\ &\equiv 6 \tan 3x \sec^2 3x. \end{aligned}$$

With an input of $x = \frac{1}{18}\pi$, we have $3x = \frac{1}{6}\pi$. The exact values we need, therefore, are $\tan \frac{1}{6}\pi = \sqrt{3}/3$ and $\cos \frac{1}{6}\pi = \sqrt{3}/2$. Reciprocating the latter gives $\sec \frac{1}{6}\pi = 2/\sqrt{3}$. This produces

$$\frac{dy}{dx} = 6 \cdot \tfrac{\sqrt{3}}{3} \cdot \left(\tfrac{2}{\sqrt{3}}\right)^2 = \tfrac{8\sqrt{3}}{3}.$$

1630. (a) Using the standard result, $\bar{X} \sim N\left(100, \frac{15^2}{40}\right)$.

(b) Calculating with the above distribution,

$$\begin{aligned} & \mathbb{P}(|\bar{X} - 100| > 5) \\ &= 1 - \mathbb{P}(95 \le \bar{X} \le 105) \\ &= 1 - 0.9649... \\ &= 0.0350 \text{ (3sf)} \end{aligned}$$

1631. We factorise, recognising the right-hand factor as a quadratic in x^2:

$$\begin{aligned} & (x^2 + 4x + 4)(x^4 + 4x^2 + 4) = 0 \\ \Longrightarrow\ & (x+2)^2 (x^2+2)^2 = 0. \end{aligned}$$

The double factor $(x+2)^2$ gives one repeated root at $x = -2$; $(x^2+2)^2$ gives no roots, as $x^2 + 2 = 0$ has none. So, there is one in total.

1632. There are two forces on the pilot: the contact force $\mathbf{R}$ and the weight $-mg\mathbf{k}$. The pilot is accelerating with the plane, so $\mathbf{a} = (\mathbf{i} + 2\mathbf{j} + \mathbf{k})g$. So, NII is

$$\begin{aligned} & \mathbf{R} - mg\mathbf{k} = m(\mathbf{i} + 2\mathbf{j} + \mathbf{k})g \\ \Longrightarrow\ & \mathbf{R} = \big(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\big) mg \\ \Longrightarrow\ & |\mathbf{R}| = \sqrt{1^2 + 2^2 + 2^2}\, mg \\ & \equiv 3mg. \end{aligned}$$

So, the chair exerts $3mg$ N on the pilot.

1633. Assuming the domain $\mathbb{R}$, the range of e^x is $(0, \infty)$, so the range of $e^x - 1$ is $(-1, \infty)$. Since the cubing function is one-to-one and since $(-1)^3 = -1$, the range of $(e^x - 1)^3$ is also $(-1, \infty)$.

1634. (a) The sum of the interior angles is $(n-2)\pi$, so, for a hexagon, 4π. Equating this with the sum of our GP, in which $a = \frac{128}{63}\pi$,

$$\frac{\frac{128}{63}\pi(1-r^6)}{1-r} = 4\pi$$
$$\implies 128 - 128r^6 = 252 - 252r$$
$$\implies 128r^6 - 252r + 124 = 0.$$

(b) The N-R iteration is

$$x_{n+1} = x_n - \frac{128x_n^6 - 252x_n + 124}{768x_n^5 - 252}.$$

Since we know that the common ratio must be less than 1, we start with, let's say, $x_0 = 0.75$. This tends to $r = \frac{1}{2}$.

(c) Using this common ratio, the smallest angle is

$$u_6 = \tfrac{128}{63}\pi \times \tfrac{1}{2}^5 = \tfrac{4}{63}\pi.$$

1635. For $\text{f}(x)$ to be polynomial, $(2x-1)$ would need to be a factor of $4x^5 - 7x^3 + 2x + 1$. To show that this is not the case, we can use the factor theorem:

$$4x^5 - 7x^3 + 2x + 1\Big|_{x=\frac{1}{2}} = \tfrac{5}{4} \neq 0.$$

Hence, $\text{f}(x)$ cannot be polynomial.

1636. Translation by vector $a\mathbf{i}$ corresponds to replacing x by $x-a$, and translation by $b\mathbf{j}$ to replacing y by $y-b$. Hence, the new equation is $|x-a|+|y-b| = 1$.

1637. Using the binomial expansion,

$$(1+x)^3 \equiv 1 + 3x + 3x^2 + x^3.$$

Splitting up the fraction,

$$\int_0^1 \frac{35(1+x)^3}{\sqrt{x}}\,dx$$
$$= 35\int_0^1 x^{-\frac{1}{2}} + 3x^{\frac{1}{2}} + 3x^{\frac{3}{2}} + x^{\frac{5}{2}}\,dx.$$

We can now integrate term by term, giving

$$35\Big[2x^{\frac{1}{2}} + 2x^{\frac{3}{2}} + \tfrac{6}{5}x^{\frac{5}{2}} + \tfrac{2}{7}x^{\frac{7}{2}}\Big]_0^1$$
$$= 35\Big(2 + 2 + \tfrac{6}{5} + \tfrac{2}{7}\Big)$$
$$= 192, \text{ as required.}$$

1638. Since $k^2 \geq 0$, the interval $\big[-3-k^2, 3+k^2\big)$ must contain the entire set $(-2, 2]$. The intersection, therefore, is simply $(-2, 2]$.

1639. This is valid. Irrespective of the hypotheses, the critical region is the complement of the acceptance region. So, $x = 2$ is indeed in the critical region. And, if a sample statistic lies in the critical region, then there is sufficient evidence to reject the null hypothesis (at the given significance level).

1640. (a) Change in velocity over $t \in [0, 2]$ is $\Delta v = 1.2$, and over $t \in [2, 4]$ is $\Delta v = 4.8 - 1.2 = 3.6$. Since $1.2 \neq 3.6$, this is not constant acceleration.

(b) Setting up $a = kt$, we integrate to $v = \frac{1}{2}kt^2 + c$. Substituting $t = 0$ and $v = 0$ gives $c = 0$. Then $t = 2$ and $v = 1.2$ gives the model as

$$v = 0.3t^2.$$

From this, we get the correct value $v = 4.8$ at $t = 4$. So, the model is consistent. Using it, $v_6 = 0.3 \times 6^2 = 10.8$.

1641. These are linear simultaneous equations in $\sqrt{x}$ and $\sqrt{y}$. Solving for these by elimination, ① + 2 × ② gives $9\sqrt{y} = 63$. So, $y = 49$. Substituting back in, we get $x = 16$.

1642. The first value is $k = 0$, for which the equation is linear: $x = -2$ is the one root. Or, if $k \neq 0$, then we have a quadratic, which has exactly one root if $\Delta = (k+1)^2 - 4k(k+2) = 0$. Solving this quadratic for k, we get $k = -1 \pm \frac{2\sqrt{3}}{3}$.

1643. At the intersections of the line and the curve, $x = 4x - x^2$, which gives $x = 0, 3$. The gradient of the curve at those points is

$$4 - 2x\Big|_{x=0,3} = 4, -2.$$

Neither of these is normal to $y = x$, which would require gradient -1. So, the statement is false.

1644. (a) The angles subtended are $\frac{1}{9} \times 360 = 40°$, and then, multiplying by the ratios, $120°$ and $200°$.

(b) Conditioning on the first spin,

$$\tfrac{1}{9} \times \tfrac{8}{9} + \tfrac{3}{9} \times \tfrac{6}{9} + \tfrac{5}{9} \times \tfrac{4}{9} = \tfrac{46}{81}.$$

1645. This is a quadratic is e^{2x}:

$$3e^{4x} - 2e^{2x} - 1$$
$$\equiv (3e^{2x} + 1)(e^{2x} - 1).$$

1646. The inputs of the process are real numbers x.

- Firstly, we must exclude irrational numbers, which cannot be expressed as p/q for $p, q \in \mathbb{Z}$: such inputs would not generate an output. So, the domain must be restricted to $\mathbb{Q}$.
- Secondly, we must exclude zero, since 0 can be expressed as $0/1$ or $0/2$ or $0/q$ for any q: this input would generate too many outputs, and the question would be one-to-many. So, the domain must be restricted to $\mathbb{Q} \setminus \{0\}$.

1647. The equation of the trajectory of a projectile is a parabola $y = \mathrm{f}(x)$. Such a curve is symmetrical about a line $x = k$ through the vertex. Since the points at which the projectile is launched and lands are both at ground level, they are symmetrical in this line. Hence, the tangents (velocities) at those points are at the same angle to the horizontal. □

——— Alternative Method ———

Let the components of initial velocity be u_x and u_y. Vertically, returning to ground level is zero displacement:

$$\begin{aligned} v_y^2 &= u_y^2 + 2 \cdot -g \cdot 0 \\ \Longrightarrow v_y^2 &= u_y^2. \end{aligned}$$

Taking the negative root, $v_y = -u_y$. Horizontal velocity remains constant. The velocity triangle is reflected vertically, giving the same angle to the horizontal. □

1648. These have linear factors iff they have real roots. Hence, testing the discriminant:

(a) True: $\Delta = -4 < 0$,

(b) True: $\Delta = -3 < 0$,

(c) False: $\Delta = 0$.

1649. (a) Solving for intersections, we get $(7/3, 8/9)$, $(1, 2)$ for the first curve, and $(7, 4)$, $(31/3, 46/9)$ for the second. The line has gradient $1/3$; we need the negative reciprocal of this. Differentiating, the relevant gradients are

$$-1 - 2x \Big|_{x=1} = -3 \text{ and } 2x - 17 \Big|_{x=7} = -3.$$

Hence, the line is normal to the curves at $(1, 2)$ and $(7, 4)$.

(b) By Pythagoras, the shortest distance between the curves is $d = \sqrt{6^2 + 2^2} = \sqrt{40}$.

1650. The two equations are rearrangements of each other, meaning that they are the same equation. Hence, the solution is infinite: any point lying on the line $4x + 6y = 1$ satisfies both equations. So, the solution set is $\{(x, y) \in \mathbb{R}^2 : 4x + 6y = 1\}$.

1651. We find the intersections of the first two circles. Multiplying out, we have $x^2 - 2x + y^2 + 4y = -4$. Subtracting $x^2 + y^2 = 2$ gives $-2x + 4y = -6$, so $x = 2y + 3$. Substituting into the second circle, $(2y+3)^2 + y^2 = 2$, so $y = -1, -7/5$. This gives the intersections at $(1, -1)$ and $(1/5, -7/5)$. Checking these, $(1, -1)$ lies on all three circles, which proves that they are concurrent.

1652. (a) $\mathrm{f}(1) = \ln 2$. We then differentiate by the chain rule to get

$$\mathrm{f}'(x) = \frac{2x}{1 + x^2}.$$

So, $\mathrm{f}'(1) = 1$.

(b) The tangent line, using $y - y_1 = m(x - x_1)$, is

$$\begin{aligned} y - \ln 2 &= 1(x - 1) \\ \Longrightarrow y &= x - 1 + \ln 2. \end{aligned}$$

Hence, $\mathrm{g}(x) = x - 1 + \ln 2$ is the linear function which best approximates $\mathrm{f}(x)$ at $x = 1$.

1653. (a) There are two successful outcomes, LL and RR. So, $p = 2 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{8}$.

(b) There are three successful outcomes, LR, RL and NN. So, $p = 2 \times \frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$.

1654. By definition, $a^{\log_a xy} \equiv xy$. Then, using an index law, $a^{\log_a x + \log_a y} \equiv a^{\log_a x} a^{\log_a y} \equiv xy$. Hence, the expressions are equivalent, giving

$$\begin{aligned} a^{\log_a xy} &\equiv a^{\log_a x + \log_a y} \\ \Longrightarrow \log_a xy &\equiv \log_a x + \log_a y. \end{aligned}$$

This is the required index law.

1655. (a) The means transforms linearly: $2\bar{x} + 3$.

(b) The mean of x_i^2 is given by $\frac{1}{n} \sum x_i^2$. We can make this the subject of the variance formula:

$$\begin{aligned} s^2 &= \frac{\sum x_i^2 - n\bar{x}^2}{n} \\ \Longrightarrow \tfrac{1}{n} \sum x_i^2 &= s^2 + \bar{x}^2. \end{aligned}$$

1656. (a) The only horizontal force is the air resistance. So, horizontal NII is $2m = ma$: deceleration is 2 ms^{-2}. Distance travelled horizontally during the fall is then

$$s = 40 \cdot 8 + \tfrac{1}{2}(-2)8^2 = 256 \text{ m}.$$

(b) Horizontal speed at landing is $40 - 8 \times 2 = 24$ ms^{-1}. Considering a velocity triangle with 30 ms^{-1} as the hypotenuse, the greatest vertical speed is $v = \sqrt{30^2 - 24^2} = 18$ ms^{-1}.

1657. The gradient of the first line is $-\frac{3}{k}$. Hence, the gradient of the second must be $\frac{k}{3}$. This gives

$$\begin{aligned} \frac{2}{k^2 - 1} &= \frac{k}{3} \\ \Longrightarrow 6 &= k^3 - k \\ \Longrightarrow (k - 2)(k^2 + 2k + 3) &= 0. \end{aligned}$$

The quadratic factor has $\Delta = 2^2 - 4 \cdot 3 = -8 < 0$, so $k = 2$.

1658. (a) Dividing top and bottom by e^x, we have

$$\lim_{x\to\infty} \frac{1}{1+e^{-2x}}.$$

Since e^{-2x} tends to zero, the limit is 1.

(b) Multiplying top and bottom by e^x, we have

$$\lim_{x\to-\infty} \frac{e^{2x}}{e^{2x}+1}.$$

The numerator tends to 0, the denominator to 1. So, the limit is 0.

1659. This is a quadratic in 7^x:

$$\begin{aligned} & 49^x - 56\cdot 7^x + 343 = 0 \\ \Longrightarrow\ & (7^x-7)(7^x-49) = 0 \\ \Longrightarrow\ & 7^x = 7, 49 \\ \Longrightarrow\ & x = 1, 2. \end{aligned}$$

1660. Consider three consecutive terms u_n, u_{n+1}, u_{n+2} in a progression. The middle term u_{n+1} may be thought of as an average of the outer two u_n and u_{n+2}.

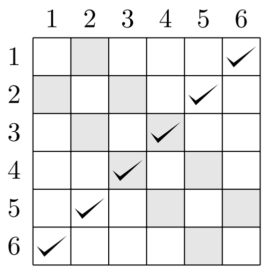
In an AP, $u_{n+2} - u_{n+1} = u_{n+1} - u_n$. This can be rearranged to $u_{n+1} = \frac{1}{2}(u_n + u_{n+2})$, which is therefore known as the *arithmetic mean.*

In a GP, $u_{n+2}/u_{n+1} = u_{n+1}/u_n$. With $u_i > 0$, this can be rearranged to $u_{n+1} = \sqrt{u_n u_{n+2}}$, which is therefore known as the *geometric mean.*

1661. The second equation, assuming $y \neq 0$, is $b = \frac{1-x}{y}$. Substituting this into the first,

$$\begin{aligned} & \left(\tfrac{1-x}{y}\right)^2 x - y = 0 \\ \Longrightarrow\ & y^3 = x(1-x)^2 \\ \Longrightarrow\ & y = x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}. \end{aligned}$$

1662. The possibility space, with restriction, is



Hence, the probability is $\frac{2}{10} = \frac{1}{5}$.

1663. Treating y as a function of x, we can differentiate implicitly by the chain rule:

$$\begin{aligned} & \frac{d}{dx}(x + y^2 - 3) = x + 1 \\ \Longrightarrow\ & 1 + 2y\frac{dy}{dx} = x + 1 \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{x}{2y}. \end{aligned}$$

— Nota Bene —

If you don't follow the implicit differentiation of y^2 above, replace y with $f(x)$. Then differentiate by the chain rule. Then replace $f(x)$ by y and $f'(x)$ by $\frac{dy}{dx}$ at the end.

1664. (a) We need to assume that the string is ① light and ② inextensible. [We are also ignoring air resistance.]

(b) The string is taut, so we can resolve along it for the system. This gives $2g = 6a$, so $a = \frac{1}{3}g$.

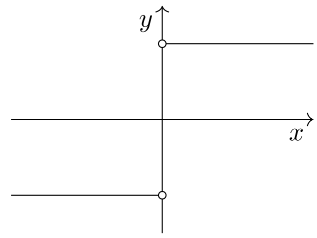
(c) NII for the hanging block is $2g - T = 2\frac{1}{3}g$, so the tension in the string is $T = \frac{4}{3}g$. This tension acts horizontally and vertically on the pulley. Pythagoras gives the overall contact force of the string on the pulley as $C = \frac{4\sqrt{2}}{3}g$. By NIII, this is also the magnitude of the force exerted on the string by the pulley.

1665. The sine function is periodic, period 2π radians. Hence, a domain of the form $[k, k+2\pi]$ constitutes one entire period. So, regardless of the value of k, the range is $[-1, 1]$.

1666. (a) This is false; counterexamples are $x = \pm 1$.

(b) This is true. Factorising gives $x(x^2+1) = 0$. But the quadratic factor has no real roots, so $x = 0$ is implied.

1667. The graph is undefined at $x = 0$. For negative x, it is $y = -1$, for positive x, it is $y = 1$.



— Nota Bene —

This function is also known, especially in coding, as the *sign function.*

1668. If there is a factorisation, it will be

$$(x^2+4)(ax^2+bx+c).$$

Equating coefficients of x^4 gives $a = 1$, of x^3 gives $b = 1$ and of x^2 gives $c = 1$. But this gives the constant term as $4 \neq -2$. Hence, there is no such factorisation.

1669. (a) Since the first two terms have even degree, they are non-negative. So, the denominator is minimised at $x = 0$, at an output of 28. The quartic is positive, so the range is $[28, \infty)$.

——— Alternative Method ———

Completing the square,

$$8x^4 + 24x^2 + 28 = 8\big(x^2 + \tfrac{3}{2}\big)^2 + 10.$$

Because of the x^2 in the bracket, the squared term cannot be zero; its minimum occurs at $x = 0$. This gives an output of 28. So, the denominator has range $[28, \infty)$.

(b) Taking the reciprocal, $f(x)$ has range $(0, 1/28]$.

1670. (a) $p = \frac{26}{52} \times \frac{25}{51} \times \frac{24}{50} \times \frac{23}{49} = \frac{46}{833}$.

(b) There are ${}^4C_2 = 6$ successful outcomes here. So, $p = 6 \times \frac{26}{52} \times \frac{25}{51} \times \frac{26}{50} \times \frac{25}{49} = \frac{325}{833}$.

1671. Algebraically, we have $f'(x) - g'(x) = 2ax + b$, where a, b are real constants. Integrating, and combining the constants of integration onto the RHS, we get $f(x) - g(x) = ax^2 + bx + c$.

The given equation $f(x) = g(x)$ is satisfied if and only if $f(x) - g(x) = 0$, which we now know occurs if and only if $ax^2 + bx + c = 0$. This is a quadratic equation, so has a maximum of two roots. □

1672. (a) i. Resolving vertically for the time of flight, $-5 = -2t - \frac{1}{2}gt^2$ gives $t = 0.82648...$. So, the range is $x = 10t = 8.26$ m (3sf).

ii. Vertical acceleration is $-\frac{4}{5}mg$. So, time of flight satisfies $-5 = -2t - \frac{1}{2} \cdot \frac{4}{5}gt^2$, giving $t = 0.90273...$. Horizontal acceleration is $-\frac{1}{5}g$. So, $x = 10t - \frac{1}{2} \cdot \frac{1}{5}gt^2 = 9.83$ m (3sf).

(b) The range with air resistance is bigger than the range without air resistance. This does not agree with reality. So, assumption of constant and equal horizontal and vertical resistances is not realistic.

——— Nota Bene ———

In fact, it is the *equality* of the resistances, rather than their *constancy*, that produces the impossible result in this particular case. The initial horizontal speed is 5 times greater than the vertical, so the horizontal resistance should be much larger.

1673. (a) A function is only invertible if it is one-to-one. Hence, $(-\infty, 2]$ or $[2, \infty)$ must give the two halves of the parabola, and $[3, \infty)$ must be its range. So, the vertex is at $(2, 3)$.

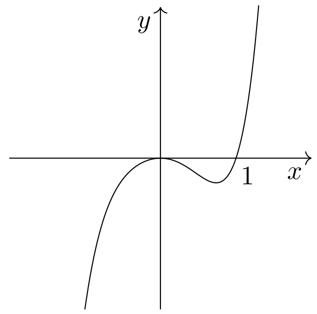
(b) Since the quadratic is monic, its equation is $f(x) = (x - 2)^2 + 3$. This gives $f(0) = 7$.

1674. (a) The expression $x^7 - x^2$ factorises as $x^2(x^5 - 1)$. By considering the quintic curve $y = x^5$ crossing the line $y = 1$, we know that the latter factor has exactly one single root, at $x = 1$. So, the full factorisation is

$$x^7 - x^2 \equiv x^2(x - 1)(\textit{quartic, no real roots}).$$

This gives the required result.

(b) Using (a), the graph $y = x^7 - x^2$ is positive polynomial of odd degree, with a double root at $x = 0$, a single root at $x = 1$ and no other real roots:

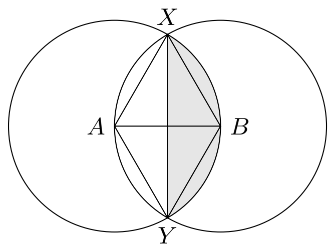


(c) We are looking for x values such that the graph is above or on the x axis. The solution set, therefore, is $\{0\} \cup [1, \infty)$.

1675. If a polynomial is of order 0 or 1, then it trivially has no points of inflection. If it has order $n \geq 2$, then the first derivative is a polynomial of order $n - 1$, and the second derivative is a polynomial of order $n - 2$. So, setting the second derivative to zero gives a polynomial of order $n - 2$, which can therefore have a maximum of $n - 2$ roots. QED.

1676. Reflection in $y = x$ switches x and y. This gives $f(y) = g(x)$. Then reflection in both $x = 0$ and $y = 0$ is a replacement of inputs (x, y) by $(-x, -y)$. Enacting this replacement, the new equation is $f(-y) = g(-x)$.

1677. We split the shaded region into two segments:



Angle XAY is 120°, so sector XAY has area $\frac{1}{3}\pi r^2$. Triangle XAY, then, has area

$$A_{\triangle} = \tfrac{1}{2}r^2 \sin 120° = \tfrac{\sqrt{3}}{4}r^2.$$

Segment XAY has area $r^2\big(\frac{1}{3}\pi - \sqrt{3}/4\big)$. Doubling,

$$A_{\text{int}} = 2r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) = r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right).$$

1678. Solving for roots,

$$\begin{aligned} & x^5 - x = 0 \\ \Longrightarrow\ & x(x+1)(x-1)(x^2+1) = 0 \\ \Longrightarrow\ & x = 0, \pm 1. \end{aligned}$$

The derivative is $f'(x) = 5x^4 - 1$. At the roots,

$$f'(0) = -1, \quad f'(\pm 1) = 4.$$

Substituting coordinates $(0,0), (\pm 1, 0)$, we have three tangent lines $y = -x$, $y = 4x \mp 4$. Hence, the required approximations are the functions

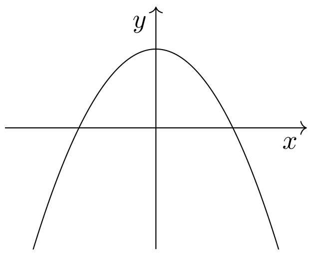
$$\begin{aligned} g_1(x) &= -x, \\ g_2(x) &= 4x - 4 \\ g_3(x) &= 4x + 4. \end{aligned}$$

1679. To do this rigorously, we divide top and bottom by x^2 before taking the limit. This gives

$$\lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{2 + \frac{1}{x^2}}.$$

As $x \to \infty$, the three inlaid fractions all tend to zero. The fraction remains well defined, with the numerator tending to 0 and the denominator to 2. So, the limit is $0/2 = 0$.

1680. The vertex must be on the y axis, so $b = 0$. The parabola must be negative, so $a < 0$. And there are distinct roots, so $c > 0$. Such parabolae are of the form



1681. Expanding binomially,

$$(2a \pm 1)^4 = 16a^4 \pm 32a^3 + 24a^2 \pm 8a + 1.$$

The $\pm$ terms cancel, leaving a biquadratic:

$$\begin{aligned} & 32a^4 + 48a^2 - 80 = 0 \\ \Longrightarrow\ & 16(a^2 - 1)(2a^2 + 5) = 0. \end{aligned}$$

The quadratic factor is never zero, so $a = \pm 1$.

1682. This doesn't hold: equilibrium isn't necessary for the resultant moment to be zero. Consider, as a counterexample, an object in freefall: the resultant moment (of the weight) is zero, yet the object is not in equilibrium.

1683. These curves are reflections of each other in $y = x$. The equation for intersections of the first curve with $y = x$ is $4x = x^2 + 1$. This has $\Delta = 17 \neq 0$, so neither curve intersects the mirror line. Hence, since they are parabolae, they don't intersect each other. So, they are not tangent.

1684. The second difference of any quadratic sequence $an^2 + bn + c$ is $2a$. Here, the first differences are $4, 6, ...$, so the second difference is 2 and $a = 1$. The first two terms give equations

$$\begin{aligned} 1 + b + c &= 12, \\ 4 + 2b + c &= 16. \end{aligned}$$

Solving these, $b = 1$ and $c = 10$. So, the ordinal formula is $u_n = n^2 + n + 10$. We now want the difference to be equal to 60. So

$$\begin{aligned} & (n+1)^2 + (n+1) + 10 - (n^2 + n + 10) = 60 \\ \Longrightarrow\ & 2n + 2 = 60 \\ \Longrightarrow\ & n = 29. \end{aligned}$$

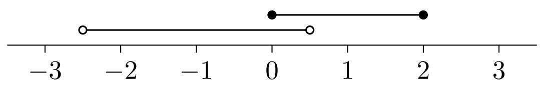
The terms are $u_{29} = 880$ and $u_{30} = 940$.

1685. Multiplying up and gathering terms,

$$\begin{aligned} & y(x^3 + 1) = x^3 - 1 \\ \Longrightarrow\ & x^3 - x^3 y = 1 + y \\ \Longrightarrow\ & x^3(1 - y) = 1 + y \\ \Longrightarrow\ & x = \sqrt[3]{\frac{1+y}{1-y}}. \end{aligned}$$

1686. (a) True,
(b) True,
(c) False; elements of $B \cap A'$ are counterexamples.

1687. On a number line, the individual sets are



Hence, the union is $(-5/2, 2]$

1688. If $2x - 1$ is to be a factor of $x^3 + 3x^2 + 16x - A$, then $x = 1/2$ must be a root. Substituting this gives $A = 1$. Taking out the factor $(2x - 1)$, we are left with $B = 4$, $C = -1$, $D = -1$.

———— Alternative Method ————

The relevant polynomial long division is

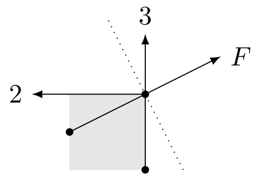
$$\begin{array}{r|l} & \quad 4x^2 \quad - x - 1 \\ \hline 2x - 1 & \quad 8x^3 - 6x^2 \quad - x + 1 \\ & -\ 8x^3 + 4x^2 \\ \hline & \qquad -\ 2x^2 \quad - x \\ & \qquad\ \ 2x^2 \quad - x \\ \hline & \qquad\qquad -\ 2x + 1 \\ & \qquad\qquad\ \ \ 2x - 1 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

1689. The curve will have a vertical asymptote wherever the denominator is zero. $\cos^2 x + \cos x + 1 = 0$ is a quadratic in $\cos\theta$, with discriminant $-3 < 0$. Hence, it has no real roots, and thus the curve has no vertical asymptotes.

    The curve also has no horizontal asymptotes, as the value of the fraction is periodic as $x \to \pm\infty$.

1690. The perpendicular bisector of points 1 and 2 is $y = x - 4$, and that of points 2 and 3 is $y = -x + 16$. Solving these simultaneously gives $P : (10, 6)$, which must be the centre of any circle passing through all four points. Testing the distances, each point is $\sqrt{20}$ from P. Hence, the quadrilateral is cyclic, as required.

1691. (a) Since there is no resultant moment, the lines of action of the three forces must be concurrent:



    (b) Having directed force $\mathbf{F}$ to achieve rotational equilibrium, we consider forces perpendicular to $\mathbf{F}$, along the dotted line. Since $\mathbf{F}$ has no component in this direction, there must be a resultant from the other two forces.

1692. By the reverse chain rule, a factor of $\frac{1}{4}$ is required:

$$\int \cos\left(4x + \tfrac{\pi}{2}\right)\,dx = \tfrac{1}{4}\sin\left(4x + \tfrac{\pi}{2}\right) + c.$$

1693. Since the two lines do not intersect, they must be parallel. Hence, $b/a = 3/2$. This gives $2b = 3a$, so $2b - 3a = 0$. Hence, $2bc - 3ac$ is also zero.

1694. (a) This is clearly true.

    (b) This is true. Since $|p| < |q|$, the vertex of the graph is at $x < 0$. Hence, in completed square form, the graph must be $y = -r(x+s)^2 + t$, where r, s, t are positive. Multiplying this out gives $b = -2rs$, which is negative.

    (c) This is clearly true.

1695. In each suit, there are nine sets of successful face values, from $\{2, 3, 4, 5, 6\}$ up to $\{10, \mathrm{J}, \mathrm{Q}, \mathrm{K}, \mathrm{A}\}$. Hence, there are $4 \times 9 = 36$ successful outcomes, out of a total of ${}^{52}\mathrm{C}_5$. This gives

$$p = \frac{36}{\frac{52!}{5! \times 47!}} = \frac{36 \times 5! \times 47!}{52!}.$$

1696. Rewriting algebraically,

$$\begin{aligned} e^{x+2} &\equiv e^x \cdot e^2 \\ &\equiv \left(10^{\ln 10}\right)^x \cdot e^2 \\ &\equiv 10^{\frac{x}{\ln 10}} \cdot e^2. \end{aligned}$$

    Starting from $y = 10^x$, we have replaced x by $x \ln 10$: this is a stretch by scale factor $\frac{1}{\ln 10}$ in the x direction. We have multiplied the outputs by e^2: this is a stretch by factor e^2 in the y direction.

    — Nota Bene —

    The scale factor $\frac{1}{\ln 10}$ could be expressed as $\log_{10} e$.

1697. (a) This is true; k is a common factor of every term, so can be taken out of the sum.

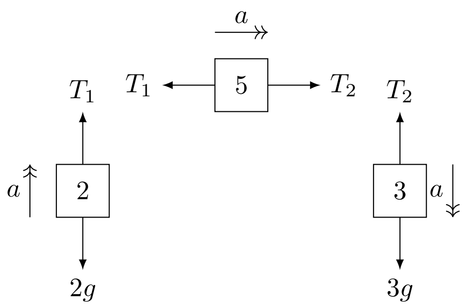
    (b) This is not true; k varies term by term in the sum, so cannot be taken out as a factor.

    (c) This is true; while n is mentioned in the sum, it is, as far as the sum is concerned, a constant. Hence, it can be taken out exactly as k could be in (a).

1698. (a) i. Model the strings as inextensible.

    ii. Model the pulleys as smooth and the strings as light.

    (b) Ignoring the vertical forces on the 5 kg mass:



    (c) The equations of motion are

$$\begin{aligned} T_1 - 2g &= 2a, \\ T_2 - T_1 &= 5a, \\ 3g - T_2 &= 3a. \end{aligned}$$

    Adding these three, both tensions cancel. We get $g = 10a$, so the acceleration is $\frac{1}{10}g$ ms^{-2}.

    — Nota Bene —

    The three masses have the same acceleration, so adding the equations of motion is the same as calculating an equation of motion for the entire system. You can do this directly.

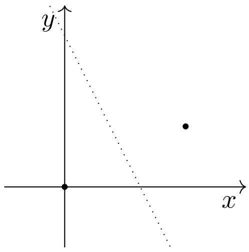
1699. The question is whether $x = \frac{\pi}{2}$ is in the domain of the function in question, i.e. whether the original function to be reciprocated is non-zero.

    (a) $\operatorname{cosec} \frac{\pi}{2} = 1$, so they intersect.

    (b) $\sec \frac{\pi}{2}$ is undefined, so they do not intersect.

    (c) $\cot \frac{\pi}{2} = 0$, so they intersect.

1700. We are not required to find the coordinates of the third vertex, so we need only note that the line passing through its two possible positions is the perpendicular bisector of the other two vertices.



This has gradient $m = -2$ and passes through the midpoint $(1, 1/2)$. So, it has equation $y = -2x + \frac{5}{2}$. We can rearrange to $2y + 4x = 5$, as required.

——— End of 17th Hundred ———